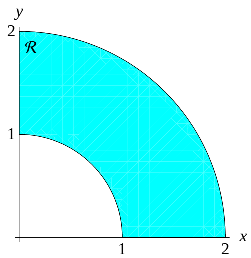
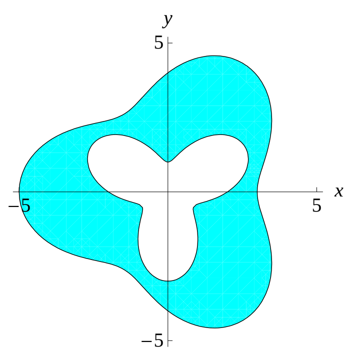
Double Integrals in Polar Coordinates

1. A flat plate is in the shape of the region $\mathcal{R}$ in the first quadrant lying between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. The density of the plate at point (x, y) is $x + y$ kilograms per square meter (suppose the axes are marked in meters). Find the mass of the plate.

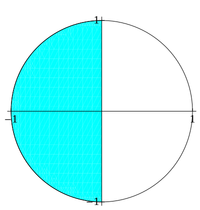


2. Find the area of the region $\mathcal{R}$ lying between the curves $r = 2 + \sin 3\theta$ and $r = 4 - \cos 3\theta$. (You may leave your answer as an iterated integral in polar coordinates.)

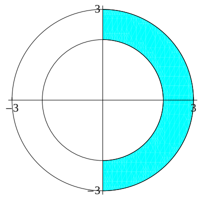


3. In each part, rewrite the double integral as an iterated integral in polar coordinates. (Do not evaluate.)

   (a) $\displaystyle\iint_{\mathcal{R}} \sqrt{1 - x^2 - y^2}\; dA$ where $\mathcal{R}$ is the left half of the unit disk.

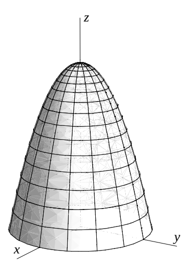


(b) $\displaystyle\iint_{\mathcal{R}} x^2 \, dA$ where $\mathcal{R}$ is the right half of the ring $4 \leq x^2 + y^2 \leq 9$.

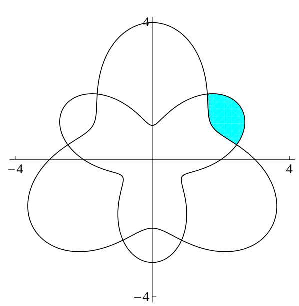


4. Rewrite the iterated integral in Cartesian coordinates $\displaystyle\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} xy \, dx \, dy$ as an iterated integral in polar coordinates. (Try to draw the region of integration.) You need not evaluate.

5. Find the volume of the solid enclosed by the xy-plane and the paraboloid $z = 9 - x^2 - y^2$. (You may leave your answer as an iterated integral in polar coordinates.)



6. The region inside the curve $r = 2 + \sin 3\theta$ and outside the curve $r = 3 - \sin 3\theta$ consists of three pieces. Find the area of one of these pieces. (You may leave your answer as an iterated integral in polar coordinates.)



When doing integrals in polar coordinates, you often need to integrate trigonometric functions. The **double-angle formulas** are very useful for this. (For instance, they are helpful for the integral in #2.)

The double-angle formulas are easily derived from the fact

$$e^{it} = \cos t + i \sin t \tag{1}$$

If θ is any angle, then

$$e^{i\theta} e^{i\theta} = e^{2i\theta}.$$

Using (1) with $t = \theta$ on the left and $t = 2\theta$ on the right, this becomes

$$\begin{aligned} (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) &= \cos 2\theta + i \sin 2\theta \\ \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta &= \cos 2\theta + i\sin 2\theta \end{aligned}$$

Equating the real parts of both sides, $\boxed{\cos^2\theta - \sin^2\theta = \cos 2\theta}$. Equating the imaginary parts, $\boxed{2\sin\theta\cos\theta = \sin 2\theta}$.

The formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ also leads to useful identities for $\cos^2\theta$ and $\sin^2\theta$:

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1 \end{aligned}$$

$$\boxed{\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)}$$

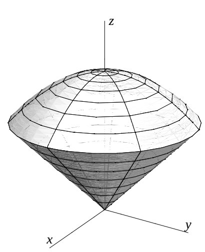
$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= (1 - \sin^2\theta) - \sin^2\theta \\ &= 1 - 2\sin^2\theta \end{aligned}$$

$$\boxed{\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)}$$

These two identities make it easy to integrate $\sin^2\theta$ and $\cos^2\theta$.

For the remaining problems, use polar coordinates or Cartesian coordinates, whichever seems easier.

7. Find the volume of the "ice cream cone" bounded by the single cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 3 - \frac{x^2}{4} - \frac{y^2}{4}$.



8. A flat plate is in the shape of the region $\mathcal{R}$ defined by the inequalities $x^2 + y^2 \leq 4$, $0 \leq y \leq 1$, $x \leq 0$. The density of the plate at the point (x, y) is $-xy$. Find the mass of the plate.

9. Find the area of the region which lies inside the circle $x^2+(y-1)^2 = 1$ but outside the circle $x^2+y^2 = 1$.